Reply to "Comment on 'Diffusion in biased turbulence'"

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(Received 30 May 2002; published 18 September 2002)

We agree with the results presented in the previous Comment [Phys. Rev. E **66**, 038301 (2002)] concerning the equality, at any time moment, of the average Lagrangian and Eulerian velocities in two-dimensional incompressible stochastic velocity fields. We show that this statistical invariance is the effect of a complex nonlinear process that determines particle trapping and a compensatory acceleration of the nontrapped particles. We discuss the possibility of developing the decorrelation trajectory method which is able to describe the process of trapping but not the statistical acceleration.

DOI: 10.1103/PhysRevE.66.038302 PACS number(s): 47.27.Qb, 05.40.−a, 52.35.Ra, 02.50.−r

We studied the effect of a constant average drift on particle diffusion in two-dimensional divergence-free stochastic velocity fields in Ref. [1]. This is essentially an analytical evaluation of the average and of the correlation of the Lagrangian velocity for a given Eulerian correlation of the potential (stream function). It is based on the decorrelation trajectory method (DTM), an approach we have recently developed [2].

In the preceding Comment [3] our result in Ref. [1] concerning the average Lagrangian velocity V^L is contested. Namely, we obtained a time-dependent average Lagrangian velocity $V^L(t)$ that evolved from the Eulerian value V_d to a smaller asymptotic value, while the Comment [3] claims the invariance of this quantity, $V^L(t)=V_d$. The arguments in Ref. [3] are based on a theorem by Lumley which is confirmed by performing a numerical simulation of the stochastic trajectories.

In the first part of this Reply we analyze the physical system considered in the numerical simulation in Ref. [3] and since it appears to be rather different from the cases we have studied until now we apply the DTM to this system. The result concerning the average Lagrangian velocity is, however, qualitatively similar to that obtained in Ref. [1]. This confirms the observation in Ref. [3] that the DTM does not provide accurate results for the average Lagrangian velocity. In the second part of our Reply we present a short analysis of the methods for studying tracer transport in stochastic velocity fields and comment on the physical significance of their basic approximations. The numerical simulation presented in the Comment [3] suggested to us the existence of a rather complex physical mechanism which determines the invariance of the average Lagrangian velocity. We show that none of the existing methods is able to describe this process although some of them obtain $V^L(t)=V_d$ and discuss the possibility of developing the DTM.

Results of the DTM for the stochastic field considered in Ref. [3]

The velocity field considered in the numerical calculation in Ref. [3] is

$$\mathbf{v}(\mathbf{x})=\frac{1}{\sqrt{N}}\sum_{n=1}^{N}[\mathbf{z}_n\cos(\mathbf{k}_n\cdot\mathbf{x})+\mathbf{y}_n\sin(\mathbf{k}_n\cdot\mathbf{x})], \qquad (1)$$

where

$$\mathbf{z}_n=\mathbf{a}_n-\frac{\mathbf{a}_n\cdot\mathbf{k}_n}{\mathbf{k}_n\cdot\mathbf{k}_n}\mathbf{k}_n, \quad \mathbf{y}_n=\mathbf{b}_n-\frac{\mathbf{b}_n\cdot\mathbf{k}_n}{\mathbf{k}_n\cdot\mathbf{k}_n}\mathbf{k}_n,$$

such that $\nabla\cdot\mathbf{v}=0$. The vectors $\mathbf{k}_n$, $\mathbf{a}_n$, and $\mathbf{b}_n$ are independent stochastic variables with Gaussian distribution, zero average, and unit variance. The Eulerian correlations of the velocity components $E_{ij}(\mathbf{r})\equiv\langle v_i(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})\rangle$ are obtained by averaging over $\mathbf{a}_n$, $\mathbf{b}_n$, and $\mathbf{k}_n$, $n=1,N$. Introducing a potential (or stream function) $\phi(\mathbf{x})$ which determines the velocity as $\mathbf{v}(\mathbf{x})=(\partial/\partial x_2,-\partial/\partial x_1)\phi(\mathbf{x})$, it can be shown that the Eulerian correlation of the potential $E(\mathbf{r})=\langle\phi(\mathbf{x})\phi(\mathbf{x}+\mathbf{r})\rangle$ is determined as

$$E(\mathbf{r})=\frac{1}{2\pi}\int d^2k\frac{1}{k^2}\exp\left(-\frac{k^2}{2}\right)\cos(\mathbf{k}\cdot\mathbf{r}), \qquad (2)$$

and after performing the integral over the angle one obtains

$$E(r)=\int_0^\infty dk\frac{1}{k}\exp\left(-\frac{k^2}{2}\right)J_0(kr), \qquad (3)$$

where $r=|\mathbf{r}|$ and J_0 is the Bessel function of the first kind. Thus the spectrum of the potential defined as the Fourier transform of the Eulerian correlation can be identified in Eq. (2) as $S(\mathbf{k})=\exp(-k^2/2)/k^2$. This spectrum is divergent in $k=0$ which means that the terms with infinite wavelengths dominate. They correspond to open contour lines of the potential. A typical realization of the potential $\phi(\mathbf{x})$ has anisotropic shape with open contour lines that form a "boulevard" along some direction. The integral in Eq. (3) is divergent for any value of r which shows that the Eulerian correlation of the potential is not defined. However, the Eulerian correlations of the velocity components $E_{ij}(\mathbf{d})$ are well defined. They can be calculated from $E(r)$ as

$$E_{11}(\mathbf{x}) = -\frac{E'(r)}{r}\frac{x_1^2}{r^2} - E''(r)\frac{x_2^2}{r^2},$$

$$E_{22}(\mathbf{x}) = -\frac{E'(r)}{r}\frac{x_2^2}{r^2} - E''(r)\frac{x_1^2}{r^2},$$

$$E_{12}(\mathbf{x}) = \frac{x_1 x_2}{r^2}\left(E''(r) - \frac{E'(r)}{r}\right),$$

and thus they are not functions of $E(r)$ but only of its first and second derivatives $E'(r)$ and $E''(r)$. These functions are well defined for all values of r and can be written as

$$E'(r) = -\frac{1}{r}\left[1 - \exp\left(-\frac{r^2}{2}\right)\right], \tag{4}$$

$$E''(r) = \frac{1}{r^2}\left[1 - \exp\left(-\frac{r^2}{2}\right)\right] - \exp\left(-\frac{r^2}{2}\right). \tag{5}$$

They are finite in $r=0$ and decay to zero when $r\to\infty$.

The DTM is based on the invariance of the Lagrangian potential and it needs the knowledge of the Eulerian correlation $E(r)$. Since the potential is defined up to a constant, arbitrary values can be extracted from the potential and consequently from its Eulerian correlation. In the case of Eq. (3) an infinite constant has to be considered. Indeed, the Eulerian correlation (3) can become finite through this procedure because its derivative with respect to r is convergent. We choose for the constant $E(0)$ and the "regularized" Eulerian correlation of the potential is

$$\mathcal{E}(r) = E(r) - E(0) = \int_0^\infty dk \frac{\exp\left(-\frac{k^2}{2}\right)}{k}[J_0(kr) - 1], \tag{6}$$

which is a well defined function of r. It is negative for all values of r, it can be approximated by $\mathcal{E}(r) \cong -r$ for $r \ll 1$ and it goes to $-\infty$ for large r as $\mathcal{E}(r) \cong -\ln(r)$. This potential defines a long-range correlated velocity field as studied in Ref. [4]. Actually it is just at the boundary between long- and short-range velocity fields.

The velocity fields we have considered in Ref. [1] are of short-range type, with Eulerian correlation that are finite in $r=0$ and decay to zero at large r. Consequently, the Eulerian velocity correlation $\langle \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}+\mathbf{r}) \rangle = -[E'(r)/r - E''(r)]$ is always negative at large r since its integral from zero to ∞ is zero. This shows the change of the direction of the velocity which is related to trajectory trapping. The corresponding correlation in Ref. [3], Eq. (6), is always positive which suggests [together with the boulevards observed in the contour plots of $\phi(\mathbf{x})$] the possibility that the trapping is not generic for this potential.

It is thus interesting to apply the DTM to this potential and to compare the results obtained by the analytical method presented in Ref. [1] with the numerical simulations in Ref. [3]. The results we have obtained are similar to those for the short-range stochastic potential, which are presented in the Fig. 2(a) in Ref. [1]. The differences are only quantitative and they show that indeed the trapping is weaker for this potential: the diffusion coefficient across $\mathbf{V}_d$ decays slower with time, the trapping regime appears at later times. In particular, the average Lagrangian velocity decays to a value which is sensibly smaller than the Eulerian average velocity as in Fig. 2(a) in Ref. [1].

In conclusion, an important qualitative difference appears in the results obtained in [1] and [3] which shows that the DTM does not provide accurate results for the average Lagrangian velocity.

Analysis of the main methods

In order to understand the source of this difference we present a short analysis of the approximations involved in the main methods used for determining the statistical properties of tracer motion described by the nonlinear Langevin equation $dx/dt = v(x)$. The basic problem consists in the evaluation of the Lagrangian velocity correlation, i.e., in the estimation of the following average of a stochastic function of a stochastic argument:

$$\langle v(0)v(x(t)) \rangle = \int dy \langle v(0)v(y)\delta[y - x(t)] \rangle. \tag{7}$$

The usual procedure is the Corrsin approximation [5]. It is based on the assumptions that the trajectories have a Gaussian distribution $P_G(y,t)$, and it neglects the correlation between the trajectories and the velocities. The average in Eq. (7) is cut in two factors $\langle v(0)v(y) \rangle$ and $\langle \delta(y - x(t)) \rangle = P_G(y,t)$. One obtains a diffusive transport of Bohm type [6] with the diffusion coefficient $D \sim V\lambda_c$, where V is the amplitude of the velocity and λ_c is its correlation length.

This is a correct result if the equation of motion would not create correlations between trajectories and the velocity field. Otherwise, the Lagrangian velocity correlations generated by the motion appear as trajectory trapping in the position and/or velocity space. The trajectories are confined in limited regions of position and/or velocity space where the velocity field is correlated and thus long-time Lagrangian correlations appear. Consequently the distribution of the displacements and/or the distribution of the Lagrangian velocity are distorted and have a non-Gaussian shape. We claim that trajectory trapping and the generation of long-time Lagrangian correlations are essentially determined by the existence of invariants of the motion.

In the case considered here (two-dimensional incompressible homogeneous static velocity fields) a spatial trapping appears which is related to the invariance of the potential $\boldsymbol{\phi}(\mathbf{x}(t))$ along the trajectory. The particles remain on the contour lines of $\phi(\mathbf{x})$ and perform a periodic motion. In consequence the transport is subdiffusive: the mean square displacement grows in time slower than linearly and the running diffusion coefficient decays to zero. The increase of the mean square displacement is determined at large t by the particles that are not yet trapped [they move on large size contour lines of $\phi(\mathbf{x})$ and have not performed a complete turn]. The spatial trapping is reflected in the distribution of the displacements which appears to be non-Gaussian [7]. The central limit theorem does not apply here because the elementary displacements $\delta\mathbf{x}=\mathbf{v}\delta t$ are not statistically independent. As far as the Lagrangian velocities are concerned, they have the same Gaussian distribution as the Eulerian velocities, because there is no trapping in velocity space.

The Corrsin approximation (as well as the improved versions) yields a diffusive transport of Bohm type and thus is not at all correct in this case. Actually this method neglects the trapping process because it relies on an assumed Gaussian distribution of the trajectories. We note that it is possible to improve the Corrsin approximation by eliminating the factorization of the average in Eq. (7) since it is possible to calculate this average exactly within the hypothesis of Gaussian distribution of displacements. However, this does not qualitatively change the results: a Bohm type diffusive transport is obtained which indirectly confirms the idea that the trajectories have a non-Gaussian distribution.

The DTM was developed having in mind the idea of maintaining the condition of invariance of the Lagrangian potential through the approximation that has to be introduced. The trajectory trapping process and subdiffusive regimes could so be obtained. To this aim, the space of realizations of the stochastic potential was divided into subensembles (S) defined by given values of the potential and of the velocity in $\mathbf{x}=\mathbf{0}$, the starting point of the trajectories. In each subensemble, there are nonzero average values of the potential and of the velocity $\langle\phi(\mathbf{x})\rangle_S=\Phi^S(\mathbf{x})$, $\langle\mathbf{v}(\mathbf{x})\rangle_S=\mathbf{V}^S(\mathbf{x})$, which are space-dependent functions determined by the Eulerian correlation of the potential. The distribution function in (S) for each one of these two quantities is a δ function in $\mathbf{x}=\mathbf{0}$ (it is a deterministic variable in this point) and at $\mathbf{x}\neq\mathbf{0}$ the distribution becomes Gaussian with an $\mathbf{x}$-dependent average and dispersion. As $\mathbf{x}$ increases from zero to infinity, the average value decreases from the value that labels the subensemble to zero, and the dispersion increases from zero up to the value corresponding to the whole statistical ensemble. The DTM is based on the following approximation concerning the average Lagrangian velocity in the subensemble:

$$\langle\mathbf{v}[\mathbf{x}(t)]\rangle_S\cong\langle\mathbf{v}[\langle\mathbf{x}(t)\rangle_S]\rangle_S=\mathbf{V}^S(\langle\mathbf{x}(t)\rangle_S). \qquad (8)$$

Thus the average Lagrangian velocity in the subensemble (S) is approximated by the average Eulerian velocity along the average trajectory $\langle\mathbf{x}(t)\rangle_S$ in (S). This means that the fluctuations of the trajectories around the average trajectory in (S) are neglected. This approximation is better in subensembles than it could be if applied in the whole set of realizations where it would be rather rough. This is due to the fact that the fluctuations of the trajectories are smaller in (S) than in the whole set of realizations. They are determined by the fluctuations of the velocity and, in the subensemble, the latter are zero in the starting point of the trajectory and become important only when the trajectory reaches large enough distances. As seen in Ref. [2] this approximation leads to a periodic average trajectory in (S). Actually, the average trajectory in (S) is a nonperiodic oscillating function of time that decays slowly to zero (since it is the average of periodic functions with periods distributed around some average value). The running diffusion coefficient $D(t)$ is represented in Ref. [2] as a weighted sum over the subensembles of the average displacement. The trapped particles do not contribute to $D(t)$ at large times because their average displacement decays to zero. A similar effect is produced by the approximate average trajectories obtained by the DTM using Eq. (8): due to an incoherent mixing of these periodic functions in the sum over the subensembles the contribution of the trapped particles is negligible in $D(t)$ at large t. Thus the DTM provides a description of the trajectory trapping and it is the first method which has obtained a subdiffusive tracer transport in such stochastic potentials, with $D(t)$ decaying algebraically to zero. A detailed study of the accuracy of this method is in progress. This qualitatively correct result is essentially due to the fact that the DTM approximation (8) preserves the invariance of the potential: the average Lagrangian potential is invariant in any subensemble (S). The Corrsin approximation has not this property and consequently it fails in describing the trapping and the subdiffusive transport.

The above physical picture becomes more complicated in the presence of a small average drift V_d. It determines an invariant average Lagrangian velocity (equal to V_d) but, as the numerical simulation in the Comment [3] suggested to us, the latter is induced by a rather nontrivial mechanism. Figure 2 in Ref. [3] shows that about half of the trajectories are trapped: they determine the peak in $x=0$ and do not contribute to the average displacement and velocity. The other part of trajectories performs an average motion with an average velocity which is about two times larger than V_d such that it compensates exactly the trapping and gives an average Lagrangian velocity equal to the Eulerian one. In general, the average Lagrangian velocity appears to be $V^L=n_f(t)V'$ where n_f is the fraction of nontrapped trajectories and V' is their average velocity. Since $n_f(t)$ is a time-dependent function $[n_f(0)=1$ and, at large time, it decays asymptotically to a smaller value $n_{fa}<1]$, the average velocity V' is also time dependent and increases according to $V'(t)=V_d/n_f(t)$ such to ensure the equality of the Lagrangian and Eulerian averages $V^L(t)=V_d$ at any time. Thus, as time increases the number of non-trapped particles decreases but their average velocity increases such that the average Lagrangian velocity is invariant. A statistical acceleration appears which increases the average Lagrangian velocity of the nontrapped particles from V_d to V_d/n_{fa}.

Corrsin approximation gives correct results for the distribution of the Lagrangain velocity $P(\mathbf{v},t)=\langle\delta(\mathbf{v}-\mathbf{v}(\mathbf{x}(t)))\rangle$. One can write this probability as

$$P(\mathbf{v},t)=\int d\mathbf{y}\langle\delta[\mathbf{v}-\mathbf{v}(\mathbf{y})]\delta[\mathbf{y}-\mathbf{x}(t)]\rangle, \qquad (9)$$

and with the hypothesis of statistical independence of trajectories and velocities which allow the factorization of the average, this equation gives the Eulerian distribution at any t, $P(\mathbf{v},t)=P_G(\mathbf{v})=\langle\delta(\mathbf{v}-\mathbf{v}(\mathbf{y}))\rangle$. In particular, the average Lagrangian velocity equals the Eulerian average

$V^L(t) = V_d$. But this result appears as a trivial consequence (almost postulated) of the approximation involved in this method and does not contained the above rather subtle effect revealed by the numerical simulation in Ref. [3].

The decorrelation trajectory method determines the asymptotic average Lagrangian velocity as $V^L = n_f(t) V_d < V_d$. Only the nontrapped particles contribute to the average Lagrangian velocity and they have an average velocity equal to the Eulerian one. The approximation (8) leads to this result for any stochastic velocity field with Eulerian correlations that decay to zero at large distances since $\mathbf{V}^S(\mathbf{x}) \to \mathbf{V}_d$ at $\mathbf{x} \gg 1$. Thus the DTM describes the trajectory trapping but does not yield the statistical acceleration observed in Ref. [3]. This is a consequence of neglecting the fluctuations of the trajectories in (S). They should determine a supplementary term on the right-hand side of Eq. (8). The fluctuations of the trajectories are determined by the fluctuations of the velocity in (S) and as the correlation of the latter increases with the distance it is expected that the correction so introduced in Eq. (8) should not decay to zero for the average trajectories that escape to ∞ and should contribute to the asymptotic average velocity yielding a value $V' > V_d$, as obtained in Ref. [3]. Thus the acceleration of the nontrapped particles should be a nonlinear effect determined by the fluctuations of the trajectories in (S).

One possibility of taking into account the fluctuations of the trajectories in (S) is to use a Corrsin approximation in each subensemble. This amounts to replacing Eq. (8), the basic approximation of the DTM, by

$$\langle \mathbf{v}(\mathbf{x}(t)) \rangle_S \cong \int d\mathbf{y} \cdot \mathbf{V}^S(\mathbf{y}) P_G(\mathbf{y} - \langle \mathbf{x}(t) \rangle_S). \tag{10}$$

It appears as an improved approximation since Eq. (8) can be obtained from Eq. (10) by neglecting the fluctuations of the trajectories in (S), i.e., by putting $\delta[\mathbf{y} - \langle \mathbf{x}(t) \rangle_S]$ instead of the Gaussian distribution $P_G[\mathbf{y} - \langle \mathbf{x}(t) \rangle_S]$. We have shown that this approximation determines an asymptotic average Lagrangian velocity equal to V_d, and thus seems to improve the result. But it does not give the image presented in Ref. [3]. The asymptotic value $V^L(t) \to V_d$ does not appear due to the acceleration of the nontrapped particles but due to the asymptotic release of all trajectories that gives $n_f(t) \to 1$. Moreover the approximation (10) does not yield the subdiffusive behavior of the transport at $V_d = 0$. The spreading of the trajectories introduced in Eq. (10) actually eliminates progressively the invariance of the Lagrangian potential and thus the trapping of the trajectories. Consequently, a finite asymptotic diffusion coefficient of Bohm type was obtained for $V_d = 0$. Thus, the improved approximation (10) actually spoils the DTM qualitatively correct results concerning the correlation of the Lagrangian velocity. These calculations determine more precisely the source of the nonlinear acceleration of the nontrapped trajectories. It is actually produced by the correlation of the fluctuations of the trajectories in the subensemble with the fluctuations of the potential, which are neglected in Eq. (10).

The conclusion of this analysis is that the description of the average motion in a two-dimensional incompressible velocity field is an open problem that could not be described by the existing methods. The average Lagrangian velocity is known from general arguments (Lumley theorem), which show that it is equal to the Eulerian average velocity V_d at any time. However, as suggested by the numerical simulations in Ref. [3], there is a rather complex nonlinear process that determines this statistical invariance. It implies trapped particles and a nonlinear statistical acceleration induced by V_d. The Corrsin approximation eliminates the trapping process and thus is not suitable for this problem. The DTM succeeds in describing trajectory trapping and the subdiffusive transport but not the nonlinear acceleration. It seems that the DTM could be developed to include this process by taking into account the fluctuations of the trajectories in the subensembles. We have shown that any attempt of developing an analytical method for describing this kind of stochastic motion has to ensure the invariance of the Lagrangian potential.

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